Math 2010

§ Differentiablity

For f : one variable function, f is differentiable at x_0 iff

$$\lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h} = f'(x_0) \quad \text{exists.}$$

$$\Leftrightarrow \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} = f'(x_0)$$

$$\Leftrightarrow \lim_{x \to x_0} \frac{f(x)-\left(f'(x_0)(x-x_0)+f(x_0)\right)}{x-x_0} = 0$$

Here $f'(x_0)(x-x_0)+f(x_0)$ is the linear approximation of $f(x)$ near x_0.

For two variables function $f(x,y)$, we prove the following fact:

If $f(x,y)$ is differentiable at (x_0,y_0), then

- $\frac{\partial f}{\partial x}(x_0,y_0)$, $\frac{\partial f}{\partial y}(x_0,y_0)$ exist.

- The linear approximation of f near (x_0,y_0) is

$$z = \left(\frac{\partial f}{\partial x}(x_0,y_0)\right)(x-x_0) + \left(\frac{\partial f}{\partial y}(x_0,y_0)\right)(y-y_0) + f(x_0,y_0)$$

However, $\frac{\partial f}{\partial x}(x_0, y_0)$, $\frac{\partial f}{\partial y}(x_0, y_0)$ exist

$\not\Rightarrow$ f is differentiable at (x_0, y_0).

Counter example: $f(x,y) = \begin{cases} r\cdot\theta & \text{for } \theta \in [0,\pi) \\ -r(\theta-\pi) & \text{for } \theta \in [\pi, 2\pi) \end{cases}$

where $x = r\cos\theta$, $y = r\sin\theta$

(check that : $\partial_x f(0,0)$, $\partial_y f(0,0)$ exist, but f is not differentiable at $(0,0)$)

§ Chain Rule:

Let $\begin{cases} f(x,y) \\ g(x,y) \end{cases}$ be two functions of two variables. Recall that:

- $\partial_x (f + cg) = \partial_x f + c\,\partial_x g$ for any $c \in \mathbb{R}$
- $\partial_x (f \cdot g) = \partial_x f\, g + f\, \partial_x g$
- $\partial_x \partial_y f = \partial_y \partial_x f$

Now, suppose that $r(t) = (x(t), y(t))$ is a smooth curve.

$$\left.\frac{d}{dt}\big(f(x(t), y(t))\big)\right|_{t=t_0} = ?$$

By definition,

$$\frac{d}{dt}\left(f(x(t),\, y(t))\right)\Big|_{t=t_0} = \lim_{t\to t_0} \frac{f(x(t)\, y(t)) - f(x(t_0)\, y(t_0))}{t-t_0}$$

$$= \lim_{t\to t_0}\left[\underbrace{\left(\frac{f(x(t)\, y(t)) - f(x(t_0),\, y(t))}{t-t_0}\right)}_{(\mathrm{I})} + \underbrace{\left(\frac{f(x(t_0),\, y(t)) - f(x(t_0),\, y(t_0))}{t-t_0}\right)}_{(\mathrm{II})}\right]$$

$$(\mathrm{II}) = \lim_{t\to t_0}\left(\frac{f(x(t_0),\, y(t)) - f(x(t_0),\, y(t_0))}{y(t)-y(t_0)}\right)\cdot\left(\frac{y(t)-y(t_0)}{t-t_0}\right)$$

$$= \partial_y f\,(x(t_0),\, y(t_0))\;\; y'(t_0)$$

$$(\mathrm{I}) = \lim_{t\to t_0}\left(\frac{f(x(t),\, y(t)) - f(x(t_0),\, y(t))}{x(t)-x(t_0)}\right)\cdot\left(\frac{x(t)-x(t_0)}{t-t_0}\right)$$

$$= \partial_x f\,(x(t_0),\, y(t_0)\cdot x'(t_0)$$

So we have the following <u>Chain Rule</u>:

$$\frac{d}{dt} f(x(t),\, y(t)) = \frac{\partial}{\partial x} f\,(x(t),\, y(t))\;\; x'(t) + \frac{\partial}{\partial y} f(x(t),\, y(t))\cdot y'(t).$$

Now, suppose we have

$$x = x(r,s)$$
$$y = y(r,s) \,. \quad r,s \text{ are two new variable}$$

Then we can differentiate

$$\frac{\partial}{\partial r} f(x(r,s),\, y(r,s)) \Big|_{(r,s)=(r_0,s_0)}$$

$$= \lim_{r \to r_0} \frac{f(x(r,s_0)\; y(r,s_0)) - f(x(r_0 s_0) - y(r_0 s_0))}{r - r_0}$$

$$= \lim_{r \to r_0} \left(\frac{f(x(r,s_0) \,.\, y(r,s_0)) - f(x(r_0 s_0) \,.\, y(r,s_0))}{r - r_0} \right) \text{ —— (I)}$$

$$+ \lim_{r \to r_0} \left(\frac{f(x(r_0 s_0)\; y(r,s_0) - f(x(r_0 s_0) ,\, y(r_0 s_0))}{r - r_0} \right) \text{ —— (II)}$$

$$(\mathrm{I}) = \lim_{r \to r_0} \left(\frac{f(x(r,s_0)\; y(r,s_0) - f(x(r_0 s_0) ,\, y(r s_0))}{x(r s_0) - x(r_0 s_0)} \right) \cdot \left(\frac{x(r,s_0) - x(r_0 s_0)}{r - r_0} \right)$$

$$= \frac{\partial f}{\partial x} (x(r_0 s_0) .\, y(r_0 s_0)) \cdot \frac{\partial}{\partial r} x(r_0 s_0)$$

Similarly,

$$(\mathrm{II}) = \frac{\partial f}{\partial y} (x(r_0 s_0) ,\, y(r_0 s_0)) \; \frac{\partial}{\partial r} y(r_0 s_0)$$

So we have the following Chain Rule formula:

$$\frac{\partial}{\partial r} f = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial r}$$

Similarly, $\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s}$

Example: Let $f(x,y) = x^2 - y^2$

Take $\begin{cases} x = r+s \\ y = r-s \end{cases} \Rightarrow r = \frac{1}{2}(x+y),\ s = \frac{1}{2}(x-y)$

So we have $f(x,y) = 4r \cdot s$

$$\frac{\partial f}{\partial r} = 4s,\quad \frac{\partial f}{\partial s} = 4r$$

Check it by using Chain Rule:

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$= 2x \cdot 1 - 2y \cdot 1$$

$$= 4s$$

and $\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s}$

$$= 2x \cdot 1 + 2y \cdot 1$$

$$= 4r.$$

Application: • Implicit Differentiation

Let $F(x,y)=0$ be a implicit function (That means we cannot write $y=f(x)$ easily in this case).

$$Q:\ \frac{dy}{dx}=?$$

By using the Chain rule (with one variable).

$$x=t,\quad y=f(x)=f(t).$$

$$\Rightarrow \frac{d}{dt}F(x(t),y(t)) = \partial_x F(x(t),y(t))\cdot x'(t) + \partial_y F(x(t),y(t))\cdot y'(t)$$

$$= \partial_x F(x,y) + \partial_y F(x,y)\cdot \frac{dy}{dx}$$

$$=0$$

So we have $\dfrac{dy}{dx} = -\dfrac{\partial_x F}{\partial_y F}$

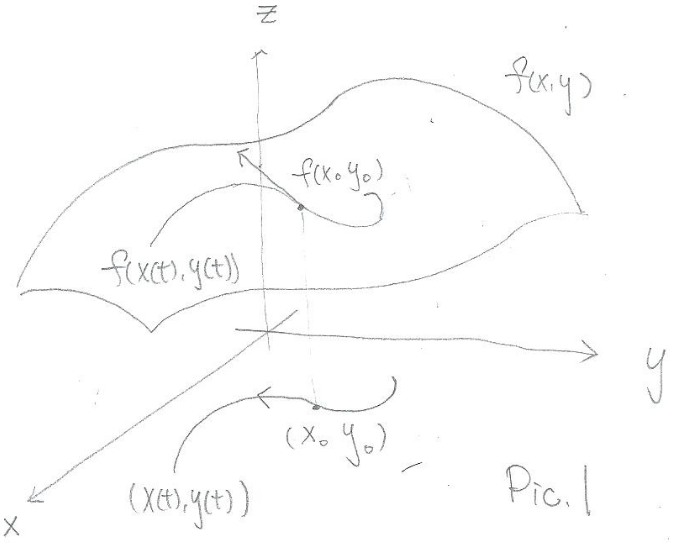
- Directional differentiation.

Let $(x(t),y(t))$ be a curve on $\mathbb{R}^2$, $x(0), y(0)=(x_0, y_0)$

The geometric meaning of

$$\frac{d}{dt}f(x(t)\, y(t))\Big|_{t=0}$$

is the directional differentiation of f along $(x(t),y(t))$ at (x_0, y_0). See Pic. 1.



Pic. 1

Example: Let $(x(t), y(t)) = \left(\frac{t}{\sqrt{2}}, \frac{t}{\sqrt{2}}\right)$, then

$$\left.\frac{df}{dt}\right|_{t=0} = \left.\frac{\partial f}{\partial x}\right|_{(0,0)} \cdot \frac{1}{\sqrt{2}} + \left.\frac{\partial f}{\partial y}\right|_{(0,0)} \cdot \frac{1}{\sqrt{2}}$$

is the directional differentiation along $x=y$.

In General, let $\vec{u} = (u_1, u_2)$ be a unit vector. We define

$$D_{\vec{u}} f := \lim_{s\to 0} \frac{f(x_0+su_1, y_0+su_2) - f(x_0, y_0)}{s}.$$

Notice that:

$$\text{RHS} = \lim_{s\to 0} \frac{f(x_0+su_1, y_0+su_2) - f(x_0, y_0+su_2)}{s\cdot u_1} \cdot u_1 \quad \left(\text{if } \begin{array}{l} u_1 \neq 0 \\ u_2 \neq 0 \end{array}\right)$$

$$+ \lim_{s\to 0} \frac{f(x_0, y_0+su_2) - f(x_0, y_0)}{s u_2} u_2$$

$$= \frac{\partial f}{\partial x}(x_0, y_0) \cdot u_1 + \frac{\partial f}{\partial y}(x_0, y_0)\, u_2$$

providing $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ are continuous near (x_0, y_0).

$$= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)\Big|_{(x_0, y_0)} \cdot (u_1, u_2)$$

So we have

$$D_{\vec{u}} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \cdot \vec{u}.$$

Def: We call this vector value function

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) := \nabla f \quad \text{gradient of } f.$$

Relationship of Gradient and Level curve

Recall that, for generic h in the image of f, we have

$$S_h = \{ (x, y) \mid f(x, y) = h \} \quad \text{is a curve.}$$

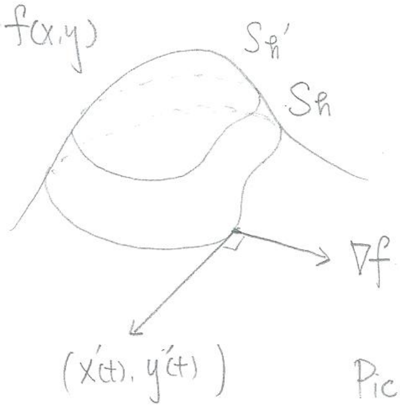
If we parametrized this curve by $(x(t), y(t))$, we have

$$\frac{d}{dt} f(x(t), y(t)) = 0$$

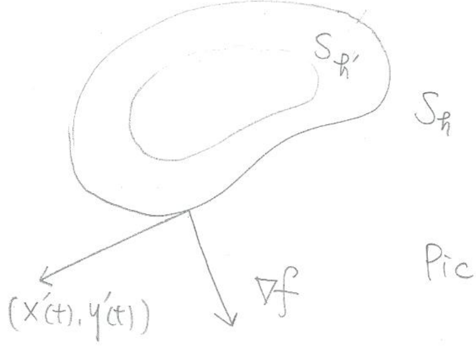
So the directional derivative of f along $(x'(t), y'(t))$ at $(x(t), y(t))$ is zero for all t.

So $\nabla f(x(t), y(t)) \cdot (x'(t), y'(t)) = 0$

$\Rightarrow \nabla f \perp (x'(t), y'(t))$ at $(x(t), y(t))$ See Pic. 2
Pic 3



Pic. 2



Pic. 3

∇f can be regarded as the direction that f changes fastest.

<u>Properties of ∇f:</u>

- (Linearity) $\nabla(f + cg) = \nabla f + c\nabla g$
- (Product rule) $\nabla(fg) = f\nabla g + g\nabla f$.

[Quotient rule $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$]